Notes on Path Integrals

Path integrals are integrals of scalar functions defined on a curve in two or three dimensions. The standard integral in a single variable x is a special case, where the curve is a portion of the x axis. Other curves can lead to much more complicated integrals.

Path Integral Concept

Let ds represent the length of a little bit of a curve C at any point. If we have the linear density $f(x,y,z)$ of some kind of stuff Q (in units of stuff per unit length), then the total amount of stuff on the curve is

$$Q = \int_C f(x,y,z)\,ds. \tag{1}$$

Evaluating Scalar Path Integrals

There is just one method for evaluating path integrals of scalar functions. The curve, integrand, and differential must all be expressed in terms of a parameter t that marks the position of points on the curve. The direction of the curve does not matter.

The challenge in finding a formula for scalar path integrals is knowing how to write ds in terms of the parameter used to identify the curve. Suppose we have a curve that is given by a position vector $\tilde{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$ on an interval $a \le t \le b$. We can find the length of the curve by thinking of $\tilde{\mathbf{r}}(t)$ as representing motion. The velocity vector is the time derivative $d\tilde{\mathbf{r}}/dt$ and the speed is the magnitude of this vector. The distance traveled in a small bit of time dt is then given by the product of velocity with time, leading to the integral

$$L = \int_a^b \left\| \frac{d\tilde{\mathbf{r}}}{dt} \right\| dt. \tag{2}$$

But we can also identify the length from (1) by taking $f = 1$ because the integral of 1 is always the size of the region. Comparing (1) and (2) yields a formula for the differential:

$$ds = \left\| \frac{d\tilde{\mathbf{r}}}{dt} \right\| dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt. \tag{3}$$

Most path integrals of scalar functions result in integrals that cannot be computed by hand. One useful special case is when the curve is a portion of any circle of radius a.

- If a circle of radius a in some plane is parameterized by an angle θ, measured from any reference radius in either direction, then the differential is simply $ds = a\,d\theta$.

Example

Suppose we want to calculate the average value of y on the curve $y = x^2$ from $(0,0)$ to $(1,1)$. We can identify the parameter t as the x coordinate, giving us the parameterization

$$x = t, \quad y = t^2, \qquad 0 \le t \le 1.$$

Then

$$ds = \sqrt{1 + (2t)^2}\,dt = \sqrt{1 + 4t^2}\,dt.$$

So the average of y is

$$\bar{y} = \frac{\int_0^1 t^2\sqrt{1+4t^2}\,dt}{\int_0^1 \sqrt{1+4t^2}\,dt}.$$

These integrals can be calculated using a trigonometric substitution, but the resulting integrals are still difficult. Using Wolfram Alpha, we obtain $\bar{y} = .60634/1.4789 = 0.4100$. Similarly, the average of x is

$$\bar{x} = \frac{\int_0^1 t\sqrt{1+4t^2}\,dt}{\int_0^1 \sqrt{1+4t^2}\,dt} \approx 0.5736.$$