Derivatives for Scalar Functions of Two or More Variables

CONCEPTS

Partial Derivative

Algebraic: The partial derivative f_x for a function $f(x, y, \ldots)$ is obtained by differentiating with respect to x while holding the other variables constant.

Geometric: f_x is the slope on a graph of f against x, with all of the other variables replaced by fixed numerical values. If $z = f(x, y)$, then f_x is the (upward) rate at which the surface $z = f$ slopes when looking in the $\vec{\mathbf{i}}$ direction from the point (x, y).

Gradient

Algebraic: The gradient of a function is the vector of partial derivatives:

$$\vec{\nabla} f(x, y, z) = \langle f_x, f_y, f_z \rangle .$$

It is convenient to think of $\vec{\nabla}$ as a vector of derivative operators:

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle .$$

This idea will allow us to use $\vec{\nabla}$ in dot and cross products to define other derivatives.

Geometric: The vector $\vec{\nabla} f(x, y, z)$ is orthogonal to the level surface of f at (x, y, z).

Directional Derivative

Algebraic: The directional derivative of a function f in the $\vec{\mathbf{u}}$ direction (where $\vec{\mathbf{u}}$ is a unit vector) is

$$\left(\frac{df}{ds} \right)_{\vec{\mathbf{u}}} = f_{\vec{\mathbf{u}}} = \vec{\mathbf{u}} \cdot \vec{\nabla} f.$$

Geometric: The directional derivative of a function $f(x, y)$ in the direction $\vec{\mathbf{u}}$ is the slope of the surface $z = f(x, y)$ for an observer looking in the $\vec{\mathbf{u}}$ direction. In particular,

$$\left(\frac{df}{ds} \right)_{\vec{\mathbf{i}}} = f_{\vec{\mathbf{i}}} \equiv \frac{\partial f}{\partial x}, \qquad \left(\frac{df}{ds} \right)_{\vec{\mathbf{j}}} = f_{\vec{\mathbf{j}}} \equiv \frac{\partial f}{\partial y}.$$

The largest directional derivative magnitude is achieved in the $\vec{\nabla} f / \|\vec{\nabla} f\|$ direction, with magnitude $\|\vec{\nabla} f\|$.

APPLICATIONS

The derivative indicates local behavior–what you see if you zoom in on the graph of a function. This idea has both geometric and algebraic applications.

Linear Approximation

Small changes in a function value near a given point can be approximated by replacing the graph of the function with that of its tangent plane. If

$$\Delta f(x,y) = f(x,y) - f(x_0,y_0) \tag{1}$$

is the change in value of a function f from a given point (x_0,y_0) to a nearby point (x,y) and

$$\langle \Delta x, \Delta y\rangle = \langle x - x_0, y - y_0\rangle \tag{2}$$

is the vector that indicates the displacement from the given point to the point (x,y), then

$$\Delta f(x,y) \approx \vec{\nabla} f(x_0,y_0) \cdot \langle \Delta x, \Delta y\rangle. \tag{3}$$

This formula can be generalized to functions of more than two variables.

Tangent Planes

The plane through the point (x_0,y_0,z_0) and *normal* to the vector $\vec{\mathbf{v}}$ is given by the equation

$$\vec{\mathbf{v}} \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0. \tag{4}$$

Suppose $F(x_0,y_0,z_0) = c$. Then the point (x_0,y_0,z_0) is on the level surface $F(x,y,z) = c$. The vector $\vec{\nabla} F(x_0,y_0,z_0)$ is normal to that level surface; hence, the plane tangent to the level surface at the given point has the equation

$$\vec{\nabla} F(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0. \tag{5}$$